International Journal of Theoretical Physics, Vol. 29, No. 4, 1990

Flat Bianchi I Friedmann–Robertson–Walker Space-Times

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Received November 21, 1988; revised December 21, 1988

In a recent paper Hajj-Boutros and Sfeila obtained the field equations in general relativity for the case of Bianchi I space-time filled with a perfect fluid and solved the equations for a particular case. In the present note the complete set of solutions of these equations is obtained.

1. INTRODUCTION

For an LRS Bianchi I space-time of the form (MacCallum, 1979)

$$ds^2 = -dt^2 + A(t)\,dx^2 + B^2(t)\,(dy^2 + dz^2) \tag{1.1}$$

and energy-momentum tensor of a perfect fluid of the form

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}; \qquad u_a u^a = -1 \tag{1.2}$$

where u^a is the 4-vector velocity, p is the pressure, and μ is the mass-energy density, the field equations in general relativity can be written as

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = -\chi_0 p \tag{1.3a}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} = -\chi_0 p \tag{1.3b}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} = \chi_0 p \tag{1.3c}$$

where the dot means differentiation with respect to t and χ_0 is Einstein's gravitational constant.

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0020-7748/90/0400-0427$06.00/0

Subtracting (1.3a) and (1.3b), one has

$$\frac{\ddot{B}}{B}-\frac{\ddot{A}}{A}-\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}=0 \tag{1.4}$$

Hajj-Boutros and Sfeila (1987) used a scale transformation $d\tau = dt/B$ in the condition of isotropy of pressures (1.7) and obtained a second-order linear differential equation

$$\frac{A''}{A}=\frac{B''}{B} \tag{1.5}$$

where the prime denotes differentiation with respect to τ.

They considered the particular case of $A = B$ and showed that more general solutions of the field equations (1.3a), (1.3b), and (1.3c) can be obtained from the solutions of the particular case $A = B$.

In the present note I generalize the work of Hajj-Boutros and Sfeila (1987) by obtaining the complete set of solutions of equations (1.3).

2. SOLUTION OF THE FIELD EQUATIONS

From (1.3a) and (1.3c), we obtain

$$\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}=0 \tag{2.1}$$

Subtracting (1.4) and (2.1), we find

$$\frac{\ddot{A}}{A}+2\frac{\dot{A}\dot{B}}{AB}=0 \tag{2.2}$$

Setting $\dot{A}B^2 = u$, equation (2.2) reduces to $\dot{u} = 0$, i.e.,

$$\dot{A}B^2 = k_1 \tag{2.3}$$

where k_1 is a constant.

Eliminating B from equations (2.1) and (2.3), we obtain

$$\frac{d}{dt}\left(\frac{\ddot{A}}{\dot{A}^2}\right)+\frac{\ddot{A}}{\dot{A}A}=0 \tag{2.4}$$

Setting $A\ddot{A}/\dot{A}^2 = Z$ in (2.4), we obtain

$$\dot{Z}=0, \qquad \text{i.e.,} \qquad \frac{A\ddot{A}}{\dot{A}^2}=k_2 \tag{2.5}$$

where k_2 is a constant.

Putting $\dot{A}/A^{k_2} = \psi$ in (2.5), we obtain

$$\dot{\psi} = 0, \qquad \text{i.e.,} \qquad \dot{A} = k_3 A^{k_2} \tag{2.6}$$

where k_3 is a constant.

Now integrating (2.6), we get

$$A = [(1-k_2)(k_3 t + k_4)]^{1/(1-k_2)} \tag{2.7}$$

where k_4 is a constant.

Substituting A from equation (2.7) into equation (2.3) gives

$$B = \left(\frac{k_1}{k_3}\right)^{1/2} [(1-k_2)(k_3 t + k_4)]^{-k_2/[2(1-k_2)]} \tag{2.8}$$

Substituting A and B from equations (2.7) and (2.8) into equation (1.3c), we obtain

$$p = \frac{k_5}{\chi_0 (k_4 + k_3 t)^2} \tag{2.9}$$

where

$$k_5 = \frac{k_2 k_3^2 (k_2 - 4)}{4(1-k_2)^2} \tag{2.10}$$

3. CONCLUSIONS

In summary, I have obtained complete solutions of the field equations in general relativity obtained by Hajj-Boutros and Sfeila (1987) for the case of a Bianchi I space-time filled with a perfect fluid, which are given by equations (1.3). Solutions are given by the expressions (2.7)-(2.9).

For $k_2 = -2$, $k_3 = k_1$, expressions (2.7) and (2.8) show that $A = B$, which particular case was considered by Hajj-Boutros and Sfeila (1987).

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